Exact Solutions of Dirac Equation on (1 + 1)-Dimensional Spacetime Coupled to a Static Scalar Field

F. Darabi · S.K. Moayedi · A.R. Ahmadi

Abstract We use a generalized scheme of supersymmetric quantum mechanics to obtain the energy spectrum and wave function for Dirac equation in (1 + 1)-dimensional spacetime coupled to a static scalar field.

1 Introduction

In the mathematical framework of two dimensional physics the exact solutions of Klein-Gordon and Dirac equations and their spectrum on a (1 + 1)-dimensional background is of particular interest. A lot of studies have been done on the motion of Klein-Gordon particle [1–3], as well as fermionic one [4, 5] on the (1 + 1)-dimensional manifolds.

In our previous works [6, 7], we have exactly solved the Klein-Gordon equation on a static (1 + 1)-dimensional space-time, and Dirac equation on a (1 + 1)-dimensional gravitational background, by using of the *standard* techniques of supersymmetric quantum mechanics. In this paper, we use a *generalized* scheme of supersymmetric quantum mechanics (SUSY QM) to obtain the exact solution and energy spectrum of Dirac equation coupled to a static scalar field in (1 + 1) dimensions. It is shown that in a special limiting case, the Dirac spinor obtained by the generalized SUSY is reduced to the spinor obtained by the standard SUSY.

F. Darabi (✉)
Department of Physics, Azarbaijan University of Tarbiat Moallem, Tabriz, Iran
e-mail: f.darabi@azaruniv.edu

S.K. Moayedi · A.R. Ahmadi
Department of Physics, Arak University, Arak, Iran

S.K. Moayedi
e-mail: S-MOAYEDI@araku.ac.ir

2 Massless Dirac Equation Coupled to a Static Scalar Field

On a $(1 + 1)$-dimensional Lorentzian manifold with the signature $(+1, -1)$, the massless Dirac equation coupled to a static scalar field $\phi(x)$ is written as [9]

$$[i\gamma^\mu \partial_\mu - \phi(x)]\Psi(x, t) = 0, \tag{1}$$

where the matrices γ^μ are the generators of Clifford algebra of two-dimensional flat space-time

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2}$$

Following Jackiw and Rebbi [8], we take the following representations for the γ^μ matrices

$$\gamma^0 = \sigma^1, \qquad \gamma^1 = i\sigma^3, \tag{3}$$

where σ^1, σ^3 are the Pauli spin matrices. We consider Ψ as a two component spinor

$$\Psi(x, t) = e^{-i\epsilon t} \begin{pmatrix} \Psi^{(1)}(x) \\ \Psi^{(2)}(x) \end{pmatrix}. \tag{4}$$

Now, the Dirac equation (1) becomes

$$\begin{cases} (\frac{d}{dx} + \phi(x))\Psi_n^{(1)}(x) = \epsilon_n \Psi_n^{(2)}(x), \\ (-\frac{d}{dx} + \phi(x))\Psi_n^{(2)}(x) = \epsilon_n \Psi_n^{(1)}(x). \end{cases} \tag{5}$$

If we define the operators

$$\begin{cases} A = \frac{d}{dx} + \phi(x), \\ A^\dagger = -\frac{d}{dx} + \phi(x), \end{cases} \tag{6}$$

then, (5) are written as

$$\begin{cases} A\Psi_n^{(1)}(x) = \epsilon_n \Psi_n^{(2)}(x), \\ A^\dagger \Psi_n^{(2)}(x) = \epsilon_n \Psi_n^{(1)}(x). \end{cases} \tag{7}$$

By operating $A^\dagger$ and A, from left, on both sides of the first and second equations in (7), respectively, we obtain the following supersymmetric equations

$$\begin{cases} H_1 \Psi_n^{(1)}(x) = \epsilon_n^2 \Psi_n^{(1)}(x) = E_n^{(1)} \Psi_n^{(1)}(x), \\ H_2 \Psi_n^{(2)}(x) = \epsilon_n^2 \Psi_n^{(2)}(x) = E_n^{(1)} \Psi_n^{(2)}(x), \end{cases} \tag{8}$$

where $E_n^{(1)} = E_n^{(2)} = \epsilon_n^2 > 0$ which is commonly called the *broken supersymmetry* [9]. To solve (8) one may use the standard technics of supersymmetric quantum mechanics.

3 Exact Solutions of 2D Dirac Equation by Generalized SUSY

We define the generalized operators [10–12]

$$\begin{cases} B = \frac{d}{dx} + \Phi(x), \\ B^\dagger = -\frac{d}{dx} + \Phi(x), \end{cases} \tag{9}$$

where $\Phi(x)$ is called the generalized *superpotential* and has the form

$$\Phi(x) = \phi(x) + g(x). \tag{10}$$

The approach of generalized supersymmetric quantum mechanics is to suppose that the partner Hamiltonian H_2 is not uniquely decomposed as $AA^\dagger$. One may then look for a suitable superpotential $\Phi(x)$ so that

$$\tilde{H}_2 = BB^\dagger = AA^\dagger = H_2. \tag{11}$$

The above equation means that both H_2 and $\tilde{H}_2$ have the same energy spectrum and wave function. Now, by inserting (6) and (9) in (11) we find that $g(x)$ must satisfy the following Riccati type equation

$$g'(x) + 2\phi(x)g(x) + g^2(x) = 0. \tag{12}$$

The partner Hamiltonian of $\tilde{H}_2$ is now defined as $\tilde{H}_1 = B^\dagger B$ which is related to H_1 as follows

$$\tilde{H}_1 = H_1 - 2g'(x). \tag{13}$$

Dirac equation in the presence of the generalized background scalar field $\Phi(x)$ is as follows

$$[i\gamma^\mu \partial_\mu - \Phi(x)]\tilde{\Psi}(x,t) = 0. \tag{14}$$

Now, by assuming the generalized spinor

$$\tilde{\Psi}(x,t) = e^{-i\epsilon t} \begin{pmatrix} \tilde{\Psi}^{(1)}(x) \\ \tilde{\Psi}^{(2)}(x) \end{pmatrix}, \tag{15}$$

and using (3) we obtain

$$\begin{cases} \tilde{H}_1 \tilde{\Psi}_n^{(1)}(x) = \epsilon_n^2 \tilde{\Psi}_n^{(1)}(x), \\ \tilde{H}_2 \tilde{\Psi}_n^{(2)}(x) = \epsilon_n^2 \tilde{\Psi}_n^{(2)}(x), \end{cases} \tag{16}$$

which are to be solved to construct the Dirac spinor (15). Now, we use the above formalism to solve exactly the Dirac equation for a typical example.

Consider the following superpotential

$$\phi(x) = 2\tanh(x). \tag{17}$$

By using of the Riccati equation (12), the function $g(x)$ for this superpotential is obtained

$$g(x) = \frac{3\cosh^{-4}(x)}{4\lambda + 3\tanh(x) - \tanh^3(x) + 2}, \tag{18}$$

where the real constant λ must be so chosen that the denominator does not vanish. If we now use the change of variable $y = \sinh(x)$, then the wave functions and the energy spectrum are obtained by the generalized SUSY as

$$\tilde{\Psi}_n^{(1)}(y) = \frac{2in}{1+y^2} P_n^{(-\frac{5}{2}, -\frac{5}{2})}(iy) + \frac{3P_{n-1}^{(-\frac{3}{2}, -\frac{3}{2})}(iy)}{(1+y^2)[y(3+2y^2) + (4\lambda+2)(1+y^2)^{\frac{3}{2}}]}, \tag{19}$$

$$\tilde{\Psi}_n^{(2)}(y) = (1+y^2)^{-\frac{1}{2}} P_{n-1}^{(-\frac{3}{2}, -\frac{3}{2})}(iy), \tag{20}$$

$$\epsilon_n^2 = n(4-n). \tag{21}$$

Whereas, using the standard SUSY we obtain

$$\Psi_n^{(1)}(y) = (1 + y^2)^{-1} P_n^{(-\frac{5}{2}, -\frac{5}{2})}(iy), \tag{22}$$

$$\Psi_n^{(2)}(y) = \tilde{\Psi}_n^{(2)}(y), \tag{23}$$

$$\epsilon_n^2 = n(4 - n). \tag{24}$$

It is to be noted that $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials with the integer n. Now we discuss about the allowed modes of the energy spectrum in (21).

Since one of the major features of the broken generalized supersymmetry is the positivity of the energy spectrum for the partner Hamiltonians $\tilde{H}_1$ and $\tilde{H}_2$, then the allowed values for the integer n are $n = 1, 2, 3, \ldots$ which lead to the allowed energy levels for Dirac particle. One may also show that considering the limiting condition $\lambda \to \infty$, the Dirac spinor obtained by (19), (20) is reduced to the one obtained by (22), (23).

4 Conclusion

In this letter we have shown that by obtaining the energy spectrum and the wave function corresponding to the Dirac equation (1) in the presence of superpotential (17), and using the non-uniqueness decomposition of H_2, namely (11), together with (10), (12) a generalized superpotential is obtained for which the wave functions and the energy spectrum for the Dirac equation (14) are obtained as (19), (20) and (21). It seems this procedure works as well for other potentials for which the Dirac equation (1) has analytic solution.

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